ALGEBRA-II

PERMUTATION & COMBINATION

P-1 Let p be a given prime and n any positive integer. Then the maximum power of p present in $\lfloor n$ is
$\left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + \left[\frac{n}{p^3}\right] + \ldots\ldots$ where $[\cdot] \Rightarrow$ GIF

P-2 Number of permutations of n different things taken r at a time is ${}^nP_r$ ie $\frac{\lfloor n}{\lfloor n-r}$

P-3 Number of permutations of n different things taken all at a time is $\lfloor n$.

P-4 Number of permutations of n things taken all at a time when p of them are alike of one type, q are alike of second type, r are alike of third type is

$$\frac{\lfloor n}{\lfloor p \lfloor q \lfloor r}$$

P-5 Number of combinations (selections) of n different things taking r at a time $(r<n)$ is ${}^nC_r$

$${}^nC_r = \frac{\lfloor n}{\lfloor n-r \cdot \lfloor r} \quad \& \quad {}^nC_r \times \lfloor r = {}^nP_r$$

P-6 ${}^nC_r = {}^nC_{n-r}$, ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$, $r \cdot {}^nC_r = n \cdot {}^{n-1}C_{r-1}$

$$\frac{{}^nC_r}{r+1} = \frac{{}^{n+1}C_{r+1}}{n+1} , \quad \frac{{}^nC_r}{{}^nC_{r-1}} = \frac{n-r+1}{r}$$

P-7 when n is even, maximum value of ${}^nC_r$ is ${}^nC_{n/2}$
when n is odd, maximum value of ${}^nC_r$ is ${}^nC_{\frac{n-1}{2}}$ or ${}^nC_{\frac{n+1}{2}}$

P-8 Number of ways of arranging n different things in circular arrangement is $\lfloor n-1$

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P-9 When clockwise and anticlockwise arrangements are not different, i.e. when observations can be made from both sides, the no. of circular arrangements of n different things is $\frac{\lfloor n-1}{2}$ (for example necklace, garland)

P-10 Total no. of combinations of n different things taken one or more at a time is $2^n - 1$.

$({}^nC_0 + {}^nC_1 + {}^nC_2 + \cdots + {}^nC_n = 2^n)$

P-11 Total no of selections of n things, out of which p things are alike of one type, q are alike of second type, r are alike of third type is $(p+q+r=n)$

$(p+1)(q+1)(r+1)$

atleast one thing $\Rightarrow (p+1)(q+1)(r+1) - 1$

P-12 If $N = p_1^a \times p_2^b \times p_3^c \times \cdots\cdots$, $a, b, c \ldots \in$ Non-negative integers

where $p_1, p_2, p_3 \ldots$ are different prime nos

(i) total no of divisors of $N = (a+1)(b+1)(c+1)\ldots$

(ii) Sum of all divisors of N is

$$\left(\frac{p_1^{a+1}-1}{p_1-1}\right) \times \left(\frac{p_2^{b+1}-1}{p_2-1}\right) \times \left(\frac{p_3^{c+1}-1}{p_3-1}\right) \times \cdots$$

(iii) All the divisors excluding 1 and N are called proper divisors.

(iv) Number of ways of writing N as a product of two natural nos is

$$\begin{cases} \frac{1}{2}[(a+1)(b+1)(c+1)\ldots] & \text{if } N \text{ is not a perfect square} \\ \frac{1}{2}[(a+1)(b+1)(c+1)\ldots + 1] & \text{if } N \text{ is a perfect square} \end{cases}$$

(v) N is a perfect square if $a, b, c \ldots$ all are even

(vi) N is a perfect ~~cube~~ cube if $a, b, c \ldots$ all are multiple of 3

(vii) If $N = 2^a \times 3^b \times 5^c \times \cdots$

a) If N is an even no, then $a \geq 1$, $b, c, \cdots \geq 0$

b) If N is an odd no, then $a = 0$, $b, c, d, \cdots \geq 0$

P-13 Division of $(m+n)$ distinct objects into two groups of size m and n $(m \neq n)$. This can be done is $\dfrac{\lfloor m+n}{\lfloor m \lfloor n}$. No of ways of distributing these two groups to two persons is $\dfrac{\lfloor m+n}{\lfloor m \lfloor n} \times \lfloor 2$

P-14 No. of ways of dividing $(m+n+p)$ distinct obj into three groups of the size m, n and p $(m \neq n \neq p)$ is

$$\frac{\lfloor m+n+p}{\lfloor m \lfloor n \lfloor p}$$

P-15 No. of ways of dividing $2n$ distinct items into two groups containing n items each is

$\dfrac{\lfloor 2n}{\lfloor n \lfloor n}$ (if groups have distinct identity)

$\dfrac{\lfloor 2n}{\lfloor n \lfloor n \lfloor 2}$ (if groups do not have distinct identity)

P-16 No. of ways of dividing $3n$ distinct items into three groups containing n items each is

$\dfrac{\lfloor 3n}{(\lfloor n)^3}$ (if groups have distinct identity)

$\dfrac{\lfloor 3n}{(\lfloor n)^3 \lfloor 3}$ (if groups do not have distinct identity)

P-17 No. of non-negative integral solⁿ of the eqⁿ

$x_1 + x_2 + x_3 + \cdots + x_r = n$ is ${}^{n+r-1}C_{r-1}$

No. of positive integral solⁿ of the eqⁿ

$x_1 + x_2 + \cdots + x_r = n$ is ${}^{n-1}C_{r-1}$

P-18 Multinomial Theorem

Consider the eq^n $x_1+x_2+x_3+\dots+x_r=n$

(i) if $x_1, x_2, x_3 \dots x_r \geq 1$

then no of sol^n is the coeff of x^n in

$$(x+x^2+x^3+\dots+x^n)\underbrace{(x+x^2+\dots+x^n)\dots\dots(x+x^2+\dots x^n)}_{r \text{ brackets}}$$

ie $(x+x^2+x^3+\dots)^r$

(ii) if $x_1 \geq 2,\ x_2 \geq 3,\ x_3, x_4 \dots x_r \geq 0$

then no of sol^n is the coeff of x^n in

$$(x^2+x^3+\dots x^n)(x^3+x^4+\dots+x^n)(x^0+x^1+x^2+\dots x^n)^{r-2}$$

(iii) No of sol^n of $ax_1+bx_2+cx_3=n$.

$a_1 \leq x_1 \leq b_1,\quad a_2 \leq x_2 \leq b_2,\quad a_3 \leq x_3 \leq b_3$

$$\Rightarrow \text{coeff of } x^n \text{ in } \Big[\{(x^a)^{a_1}+(x^a)^{a_1+1}+\dots+(x^a)^{b_1}\}\{(x^b)^{a_2}+(x^b)^{a_2+1}+\dots(x^b)^{b_2}\}$$
$$\{(x^c)^{a_3}+(x^c)^{a_3+1}+\dots+(x^c)^{b_3}\}\Big]$$

e.g. No of sol^n of $3x_1+2x_2+x_3=14$, $x_1, x_2, x_3 \geq 1$

$$\Rightarrow \text{coeff of } x^{14} \text{ in } \left[(x^3+x^6+\dots)(x^2+x^4+x^6+\dots)(x+x^2+x^3+\dots)\right]$$

(iv) If there are l obj of one type, m obj are of second type, n obj are of third type and so on, then no of ways of choosing r obj out of these obj is the coeff of x^r in

$$(1+x+x^2+\dots+x^l)(1+x+x^2+\dots+x^m)(1+x+x^2+\dots+x^n)$$

Note If upper limit of a variable is more than or equal to the sum required, then the upper limit of that variable can be taken as infinite.

(v) No of non-ve integral sol^n of $x_1+x_2+x_3+\dots+x_m \leq n$ is same as no of non-negative int. sol^n of $x_1+x_2+x_3+\dots+x_m+t=n$ where t is a dummy variable.

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(vi) To find the no. of sol^n of $x_1+x_2+x_3+\dots+x_n \geq n$ (where value of $x_1, x_2, \dots x_n$ are restricted), first find the no. of sol^n of $x_1+x_2+\dots+x_n \leq n-1$, & then subtract it from the total no of sol^{n}s.

P-19 Sum of all n-digit numbers formed using n digits

$$= \lfloor n-1 \, (\text{Sum of all } n \text{ digits}) \times (\underbrace{1111\dots1}_{n \text{ times}})$$

P-20 Number of diagonals in n-sided polygon is $({}^nC_2 - n)$ i.e. $\frac{n(n-3)}{2}$

P-21 Number of squares in two system of perpendicular parallel lines (when 1^{st} set contains m parallel lines and 2^{nd} set contains n parallel lines) is equal to

$$\sum_{r=1}^{m-1} (m-r)(n-r) \; ; \; (m<n).$$

P-22 Derangement

n letters are to be kept in n corresponding envelopes, no of ways in which they can be placed if none of the letter goes into its own envelope is

$$\lfloor n \left[1 - \frac{1}{\lfloor 1} + \frac{1}{\lfloor 2} - \frac{1}{\lfloor 3} + \frac{1}{\lfloor 4} \dots + (-1)^n \frac{1}{\lfloor n}\right]$$

P-23 Number of Functions

Let $X = \{x_1, x_2, x_3 \dots x_n\}$ & $Y = \{y_1, y_2, y_3 \dots y_r\}$

(i) Total no of fun^{n}s $= r^n$

(ii) Total no of one-one fun^{n}s $= \begin{cases} {}^rC_n \times \lfloor n, & r \geq n \\ 0, & r < n \end{cases}$

(iii) Total no of many-one fun^n $= \begin{cases} r^n - {}^rC_n \times \lfloor n, & r \geq n \\ r^n, & r < n \end{cases}$

(iv) Total no of constant fun^n $= r$

(v) Total no of onto fun^n $= \begin{cases} r^n - {}^rC_1 (r-1)^n + {}^rC_2 (r-2)^n - {}^rC_3 (r-3)^n + \dots & r < n \\ \lfloor r, & r = n \\ 0, & r > n \end{cases}$

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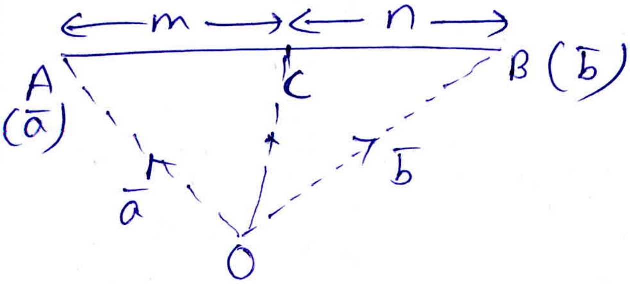
(vi) Total no of into fun^n

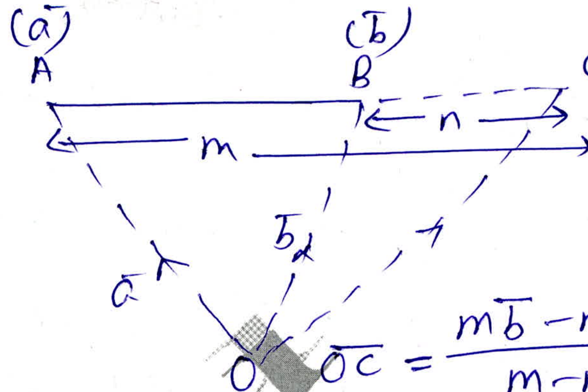
$$= \begin{cases} {}^{r}C_1 (r-1)^n - {}^{n}C_2 (r-2)^n + {}^{r}C_3 (r-3)^n + \cdots & r \le n \\ r^n & , \quad r > n \end{cases}$$

VECTORS & 3-D GEOMETRY

V-1 Section formula

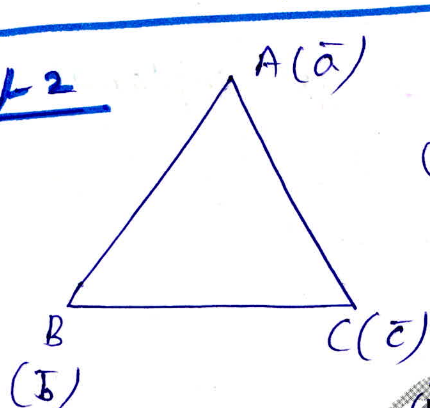


$$\overline{OC} = \frac{m\bar{b} + n\bar{a}}{m+n}$$



$$\overline{OC} = \frac{m\bar{b} - n\bar{a}}{m-n}$$

V-2



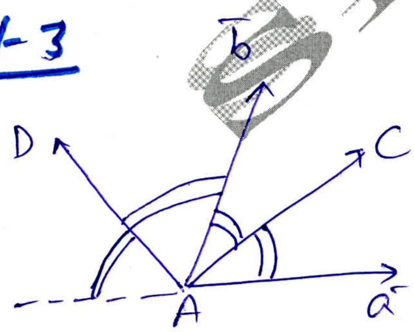
(i) Centroid $= \dfrac{\bar{a}+\bar{b}+\bar{c}}{3}$

(ii) Incentre $= \dfrac{BC\,\bar{a} + AC\,\bar{b} + AB\,\bar{c}}{AB+BC+CA}$

(iii) Orthocentre $= \dfrac{\tan A\,\bar{a} + \tan B\,\bar{b} + \tan C\,\bar{c}}{\tan A + \tan B + \tan C}$

(iv) Circumcentre $= \dfrac{\sin 2A\,\bar{a} + \sin 2B\,\bar{b} + \sin 2C\,\bar{c}}{\sin 2A + \sin 2B + \sin 2C}$

V-3



Any vector along the bisector of the angle formed by two vectors $\bar{a}$ & $\bar{b}$ is $\lambda\left(\dfrac{\bar{a}}{|a|} + \dfrac{\bar{b}}{|b|}\right)$

$AC =$ i.e. $\lambda(\hat{a} + \hat{b})$

and any vector along the external bisector

$\overline{AD} = \lambda(\hat{a} - \hat{b})$

V-4

(i) A system of vectors $a_1, a_2, a_3, \ldots a_n$ is said to linearly independent if

$m_1\bar{a}_1 + m_2\bar{a}_2 + \cdots + m_n\bar{a}_n = 0 \Rightarrow m_1 = m_2 = \cdots = m_n = 0.$

(ii) A set of vectors $a_1, a_2, a_3, \ldots a_n$ is said to be linearly dependent if there exist scalars $m_1, m_2, \ldots m_n$ not all zero. such that $m_1\bar{a}_1 + m_2\bar{a}_2 + \cdots + m_n\bar{a}_n = 0$

$\Rightarrow$ A pair of collinear vectors is linearly dependent

$\Rightarrow$ A triad of coplanar vectors is linearly dependent.

V-5 (i) Two collinear vectors are always linearly dependent.

(ii) Two non-collinear non-zero vectors are always linearly independent.

(iii) Three coplanar vectors are always linearly dependent.

(iv) Three non-coplanar non-zero vectors are always linearly independent

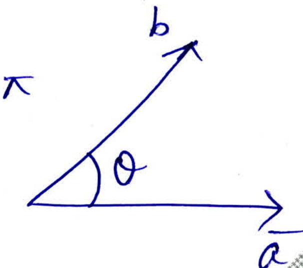
(v) More than three vectors are always linearly dependent

(vi) Three pts with position vectors $\bar{a}, \bar{b}$ & $\bar{c}$ are collinear if and only if there exist scalars x, y, z not all zero such that $x\bar{a}+y\bar{b}+z\bar{c}=0$ & $x+y+z=0$

(vii) Four pts with position vectors $\bar{a}, \bar{b}, \bar{c}$ & $\bar{d}$ are coplanar if there exist scalars x, y, z & w (sum of any two is not zero) such that $x\bar{a}+y\bar{b}+z\bar{c}+w\bar{d}=0$ &

$x+y+z+w=0$.

V-6 Dot (Scalar) Product-



$\bar{a}\cdot\bar{b} = |\bar{a}||\bar{b}|\cos\theta, \quad 0 \le \theta \le \pi$

θ is acute $\Rightarrow \bar{a}\cdot\bar{b} > 0$

θ is obtuse $\Rightarrow \bar{a}\cdot\bar{b} < 0$

θ is right angle $\Rightarrow \bar{a}\cdot\bar{b} = 0$

projection of $\bar{a}$ on $\bar{b}$ = $\dfrac{\bar{a}\cdot\bar{b}}{|\bar{b}|}$

projection of $\bar{b}$ on $\bar{a}$ = $\dfrac{\bar{a}\cdot\bar{b}}{|\bar{a}|}$

$\Rightarrow \bar{a}\cdot\bar{a} = |a||a|\cos 0° = |a|^2 = a^2 \Rightarrow \hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1$

$\Rightarrow \hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{i} = 0$

$\Rightarrow \bar{a}\cdot\bar{b} = \bar{b}\cdot\bar{a}, \quad \bar{a}\cdot(\bar{b}+\bar{c}) = \bar{a}\cdot\bar{b} + \bar{a}\cdot\bar{c}$

$\Rightarrow (\bar{a}\pm\bar{b})^2 = |\bar{a}|^2 + |\bar{b}|^2 \pm 2|\bar{a}||\bar{b}|\cos\theta$

$\Rightarrow \bar{a} = a_1\hat{i}+a_2\hat{j}+a_3\hat{k}, \; \bar{b} = b_1\hat{i}+b_2\hat{j}+b_3\hat{k} \Rightarrow \bar{a}\cdot\bar{b} = a_1b_1+a_2b_2+a_3b_3$

$\Rightarrow \bar{r} = (\bar{r}\cdot\hat{i})\hat{i} + (\bar{r}\cdot\hat{j})\hat{j} + (\bar{r}\cdot\hat{k})\hat{k}$

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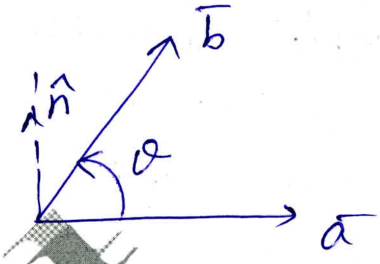
$\Rightarrow$ Angle b/w two vectors $\bar{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ & $\bar{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ is

$$\cos\theta = \frac{\bar{a}\cdot\bar{b}}{|a|\,|b|} = \frac{a_1 b_1 + a_2 b_2 + a_3 b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2} \times \sqrt{b_1^2 + b_2^2 + b_3^2}}$$

V-7 Cross (Vector) Product-



$\bar{a} \times \bar{b} = |\bar{a}|\,|\bar{b}| \sin\theta\ \hat{n}$

$|\bar{a} \times \bar{b}| = |\bar{a}|\,|\bar{b}| \sin\theta$

$\bar{a} \times \bar{b} = -\bar{b} \times \bar{a}$, $\bar{a} \times \bar{a} = 0$

$\bar{a} \times (\bar{b} + \bar{c}) = \bar{a} \times \bar{b} + \bar{a} \times \bar{c}$

$\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = 0$, $\hat{i} \times \hat{j} = \hat{k}$, $\hat{j} \times \hat{k} = \hat{i}$, $\hat{k} \times \hat{i} = \hat{j}$

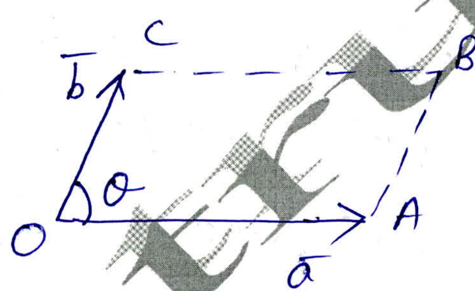
Two non-zero vectors $\bar{a}$ & $\bar{b}$ are collinear if and only if $\bar{a} \times \bar{b} = 0$

If $a = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, $\bar{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

$$\bar{a} \times \bar{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \hat{i}(a_2 b_3 - a_3 b_2) + \hat{j}(a_1 b_3 - b_1 a_3) + \hat{k}(a_1 b_2 - a_2 b_1)$$

The unit vector perpendicular to $\bar{a}$ & $\bar{b}$ is $\dfrac{\bar{a} \times \bar{b}}{|\bar{a} \times \bar{b}|}$

V-8 Area of parallelogram

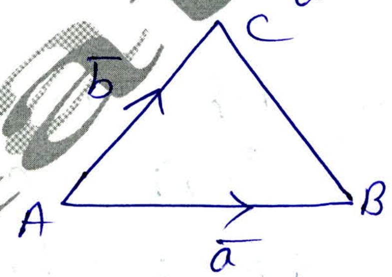


$OABC = |\bar{a} \times \bar{b}|$

$= |\bar{a}|\,|\bar{b}| \sin\theta$

Area of triangle ABC is

$\frac{1}{2} |\bar{a} \times \bar{b}|$



If $\bar{p}$ and $\bar{q}$ are diagonals of a parallelogram, its area $= \frac{1}{2} |\bar{p} \times \bar{q}|$

The ~~area~~ area of a triangle whose vertices are $A(\bar{a})$, $B(\bar{b})$ & $C(\bar{c})$ is $\frac{1}{2} |\bar{a} \times \bar{b} + \bar{b} \times \bar{c} + \bar{c} \times \bar{a}|$.

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V-9. Scalar Triple Product

$(\bar{a}\times\bar{b})\cdot\bar{c}$ denoted by $[\bar{a}\ \bar{b}\ \bar{c}]$

$(\bar{a}\times\bar{b})\cdot\bar{c} = \bar{a}\cdot(\bar{b}\times\bar{c}) = \bar{b}\cdot(\bar{c}\times\bar{a}) = \bar{c}\cdot(\bar{a}\times\bar{b})$

Here $(\bar{a}\times\bar{b})\cdot\bar{c}$ represents the volume of the parallelopiped whose adjacent sides are $\bar{a}, \bar{b}$ & $\bar{c}$.

$\bar{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, $\bar{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, $\bar{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$

$$(\bar{a}\times\bar{b})\cdot\bar{c} = [\bar{a}\ \bar{b}\ \bar{c}] = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Volume of tetrahedron ABCD is equal to $\frac{1}{6}(\overline{AB}\times\overline{AC})\cdot\overline{AD}$

$[k\bar{a}\ \bar{b}\ \bar{c}] = k[\bar{a}\ \bar{b}\ \bar{c}]$.

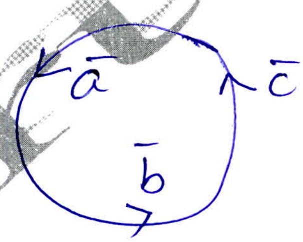
$[\bar{a}+\bar{b}\ \ \bar{c}\ \ \bar{d}] = [\bar{a}\ \bar{c}\ \bar{d}] + [\bar{b}\ \bar{c}\ \bar{d}]$

Three non-zero, non-collinear vectors $\bar{a}, \bar{b}, \bar{c}$ are coplanar if $[\bar{a}\ \bar{b}\ \bar{c}] = 0$

$[\bar{a}\ \bar{a}\ \bar{b}] = 0$, $[\bar{a}\ \bar{b}\ \bar{c}] = -[\bar{b}\ \bar{a}\ \bar{c}] = -[\bar{c}\ \bar{b}\ \bar{a}]$

$[\bar{a}\times\bar{b}\ \ \bar{b}\times\bar{c}\ \ \bar{c}\times\bar{a}] = [\bar{a}\ \bar{b}\ \bar{c}]^2$

$[\bar{a}+\bar{b}\ \ \bar{b}+\bar{c}\ \ \bar{c}+\bar{a}] = 2[\bar{a}\ \bar{b}\ \bar{c}]$



V-10 Vector Triple Product

$\bar{a}\times(\bar{b}\times\bar{c}) = (\bar{a}\cdot\bar{c})\bar{b} - (\bar{a}\cdot\bar{b})\bar{c}$

$(\bar{a}\times\bar{b})\times\bar{c} = (\bar{a}\cdot\bar{c})\bar{b} - (\bar{b}\cdot\bar{c})\bar{a}$

$$(\bar{a}\times\bar{b})\cdot(\bar{c}\times\bar{d}) = \begin{vmatrix} \bar{a}\cdot\bar{c} & \bar{a}\cdot\bar{d} \\ \bar{b}\cdot\bar{c} & \bar{b}\cdot\bar{d} \end{vmatrix}$$

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V-11 Reciprocal System of Vectors

Two system of vectors are called reciprocal system of vectors if by taking dot product we get unity.

if $a' = \dfrac{\bar{b}\times\bar{c}}{[\bar{a}\bar{b}\bar{c}]}$, $b' = \dfrac{\bar{c}\times\bar{a}}{[\bar{a}\bar{b}\bar{c}]}$, $c' = \dfrac{\bar{a}\times\bar{b}}{[\bar{a}\bar{b}\bar{c}]}$

a', b', c' are said to be the ~~ϕ~~ reciprocal system of vectors for $\bar{a}, \bar{b}$ & $\bar{c}$.

$\Rightarrow$ $a\cdot a' = b\cdot b' = c\cdot c' = 1$

$\Rightarrow$ $a\cdot b' = a\cdot c' = b\cdot a' = b\cdot c' = c\cdot a' = c\cdot b' = 0$

$\Rightarrow$ $[\vec{a}\,\vec{b}\,\vec{c}]\,[\vec{a}'\,\vec{b}'\,\vec{c}'] = 1$

$\Rightarrow$ The orthogonal triad of vectors $\hat{i}, \hat{j}$ & $\hat{k}$ is self-reciprocal.

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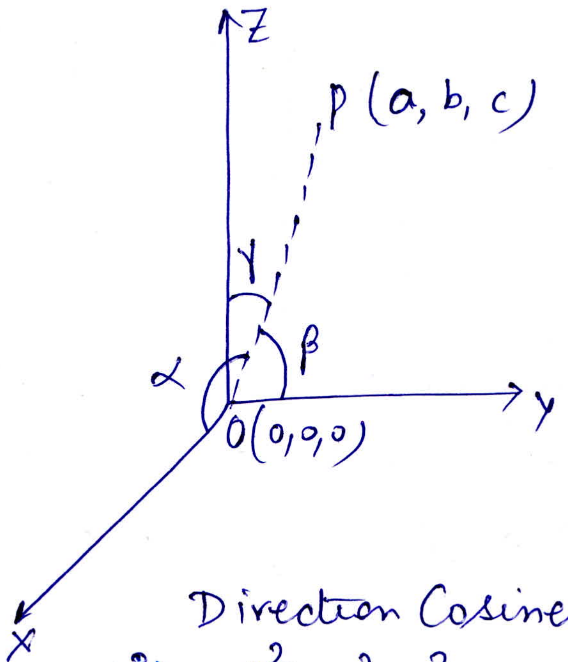
3-D

D-1 Direction Cosines and Direction Ratios



Vector $\overline{OP}$ makes angles α, β, γ with the x, y & z axes respectively.
Here, α, β, γ are called directional angles, and, $\cos\alpha$, $\cos\beta$ & $\cos\gamma$ are called direction cosines.

DC of x axis $(1,0,0)$
DC of y axis $(0,1,0)$
DC of z axis $(0,0,1)$

Direction Cosines are denoted by l, m, n.

(i) $l^2+m^2+n^2 = 1$

(ii) $\bar{r} = |\bar{r}|(l\hat{i}+m\hat{j}+n\hat{k})$ & $\hat{r} = l\hat{i}+m\hat{j}+n\hat{k}$

Let l, m & n be the DC of a vector $\bar{r}$ and a, b & c be three numbers such that a, b, c are proportional to l, m & n. Hence $\frac{l}{a} = \frac{m}{b} = \frac{n}{c} = k$ or $(l,m,n) \equiv (ka, kb, kc)$

Here, a, b & c are directional ratios.

For example, if $\left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$ are DC of a vector $\bar{r}$, then its direction ratios are $(1,-1,1)$ or $(-1,1,-1)$ or $(2,-2,2)$ or $(\lambda, -\lambda, \lambda)$....

That shows there can be an infinite number of directions ratios for a given vector, but the direction cosines are unique.

D-2 DR of a line joining two points.

$P(x_1, y_1, z_1)$ & $Q(x_2, y_2, z_2)$

$\overline{PQ} = (x_2-x_1)\hat{i} + (y_2-y_1)\hat{j} + (z_2-z_1)\hat{k}$

DR of $\overline{PQ}$ are $\langle (x_2-x_1), (y_2-y_1), (z_2-z_1) \rangle$

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$\bar{r} = a\hat{i} + b\hat{j} + c\hat{k}$

D-3 $l = \lambda a$, $m = \lambda b$, $n = \lambda c$, where $l, m, n \to DC$

$a, b, c \to DR$

$\Rightarrow l^2 + m^2 + n^2 = 1 \Rightarrow \lambda = \pm \dfrac{1}{\sqrt{a^2+b^2+c^2}}$

$\Rightarrow l = \pm \dfrac{a}{\sqrt{a^2+b^2+c^2}}$, $m = \pm \dfrac{b}{\sqrt{a^2+b^2+c^2}}$, $n = \pm \dfrac{c}{\sqrt{a^2+b^2+c^2}}$

D 4 Points to Remember

(i) Projections of $\bar{r}$ on the coordinate axes are $l|\bar{r}|, m|\bar{r}|, n|\bar{r}|$.

(ii) The projection of a segment joining points $P(x_1, y_1, z_1)$ & $Q(x_2, y_2, z_2)$ on a line with direction cosines l, m & n is $(x_2 - x_1)l + (y_2 - y_1)m + (z_2 - z_1)n$.

(iii) If l_1, m_1, n_1 & l_2, m_2, n_2 are the DC's of two concurrent lines, then the DC's of the lines bisecting the angles between them are proportional to $l_1 \pm l_2$, $m_1 \pm m_2$, $n_1 \pm n_2$.

(iv) Let l_1, m_1, n_1 & l_2, m_2, n_2 be DC's of two lines and a_1, b_1, c_1 & a_2, b_2, c_2 be their Direction Ratios.

if θ be the acute angle b/w the two lines

$\cos\theta = |l_1 l_2 + m_1 m_2 + n_1 n_2|$ or $\cos\theta = \dfrac{|a_1 a_2 + b_1 b_2 + c_1 c_2|}{\sqrt{a_1^2+b_1^2+c_1^2} \times \sqrt{a_2^2+b_2^2+c_2^2}}$

a) two lines are perpendicular if

$l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$ or $a_1 a_2 + b_1 b_2 + c_1 c_2 = 0$

b) two lines are parallel if

$\dfrac{l_1}{l_2} = \dfrac{m_1}{m_2} = \dfrac{n_1}{n_2}$ or $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{c_1}{c_2}$

D-5 Direction ~~ratio~~ cosines of line along the bisector of two given lines

If $l_1 m_1 n_1$ & $l_2 m_2 n_2$ are DC's of two lines inclined to each other at an angle θ then the DC's of the

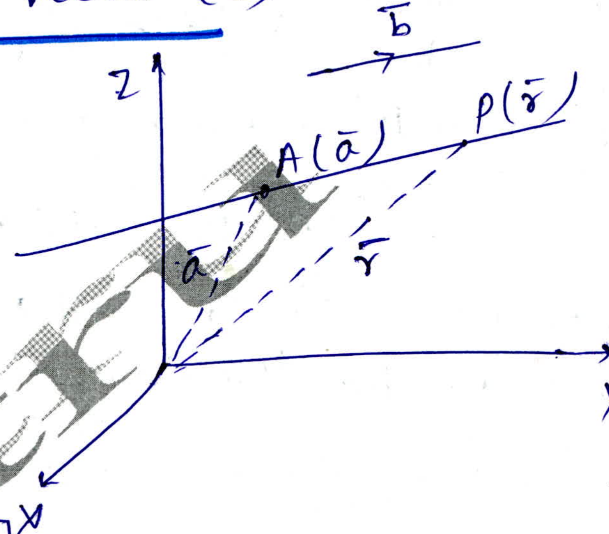
(a) internal bisector of the angle b/w the two lines are

$\dfrac{l_1 + l_2}{2\cos\theta/2}$, $\dfrac{m_1 + m_2}{2\cos\theta/2}$, $\dfrac{n_1 + n_2}{2\cos\theta/2}$

b) external bisector of the angle between the lines are

$$\frac{l_1-l_2}{2\sin\theta/2},\ \frac{m_1-m_2}{2\sin\theta/2},\ \frac{n_1-n_2}{2\sin\theta/2}.$$

D-6 Eq^n of straight line passing through a given point $(\bar{a})$ and parallel to a given vector $(\bar{b})$



$$\boxed{\bar{r}=\bar{a}+\lambda\bar{b}}$$ vector form

Cartesian form

$\bar{r}=x\hat{i}+y\hat{j}+z\hat{k}$

$\bar{a}=x_1\hat{i}+y_1\hat{j}+z_1\hat{k}$

$\bar{b}=a\hat{i}+b\hat{j}+c\hat{k}$

$$\Rightarrow \boxed{\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c}=\lambda}$$

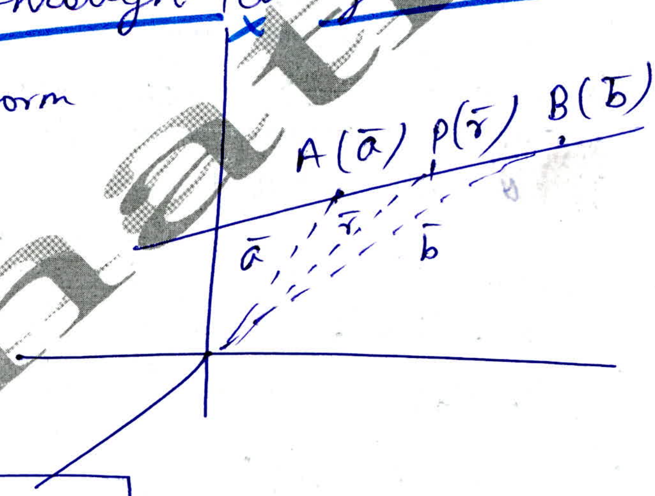
(i) Here any pt on the line can be taken as $(x_1+a\lambda,\ y_1+b\lambda,\ z_1+c\lambda)$.

(ii) Eq^n of x axis: $\frac{x-0}{1}=\frac{y-0}{0}=\frac{z-0}{0}$ or $y=0, z=0$

y axis: $\frac{x-0}{0}=\frac{y-0}{1}=\frac{z-0}{0}$ or $x=0, z=0$

z axis: $\frac{x-0}{0}=\frac{y-0}{0}=\frac{z-0}{1}$ or $x=0, y=0$.

D-7 Eq^n of line passing through two given pts $\bar{a}$ & $\bar{b}$



$$\boxed{\bar{r}=\bar{a}+\lambda(\bar{b}-\bar{a})}$$ vector form

Cartesian form

$\bar{r}=x\hat{i}+y\hat{j}+z\hat{k}$

$\bar{a}=x_1\hat{i}+y_1\hat{j}+z_1\hat{k}$

$\bar{b}=x_2\hat{i}+y_2\hat{j}+z_2\hat{k}$

$$\Rightarrow \boxed{\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}=\lambda}$$

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D-8 Angle b/w two lines

$\vec{r} = \vec{a} + \lambda \vec{b}$ & $\vec{r} = \vec{c} + \lambda \vec{d}$ is

$$\cos\theta = \frac{\vec{b} \cdot \vec{d}}{|\vec{b}|\,|\vec{d}|}$$

if $\frac{x-x_1}{b_1} = \frac{y-y_1}{b_2} = \frac{z-z_1}{b_3}$ & $\frac{x-x_2}{d_1} = \frac{y-y_2}{d_2} = \frac{z-z_2}{d_3}$

$$\cos\theta = \frac{|b_1 d_1 + b_2 d_2 + b_3 d_3|}{\sqrt{b_1^2+b_2^2+b_3^2}\,\sqrt{d_1^2+d_2^2+d_3^2}}$$

if lines are $\perp$ to each other $\Rightarrow b_1 d_1 + b_2 d_2 + b_3 d_3 = 0$

if lines are parallel to each other $\Rightarrow \frac{b_1}{d_1} = \frac{b_2}{d_2} = \frac{b_3}{d_3}$

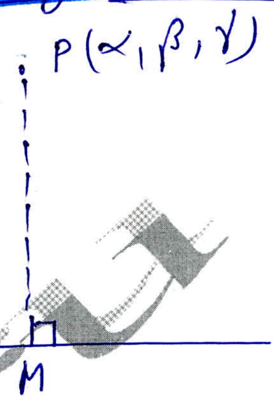
D-9 foot of perpendicular from a point on given line

line: $\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c} = \lambda$

point M : $(a\lambda + x_1,\ b\lambda + y_1,\ c\lambda + z_1)$ —(1)

DR of PM: $(x_1 + a\lambda - \alpha,\ y_1 + b\lambda - \beta,\ z_1 + c\lambda - \gamma)$

PM $\perp$ line



$\therefore a(x_1 + a\lambda - \alpha) + b(y_1 + b\lambda - \beta) + c(z_1 + c\lambda - \gamma) = 0$

$\Rightarrow$ find the value of λ and put in (1) to get point M.

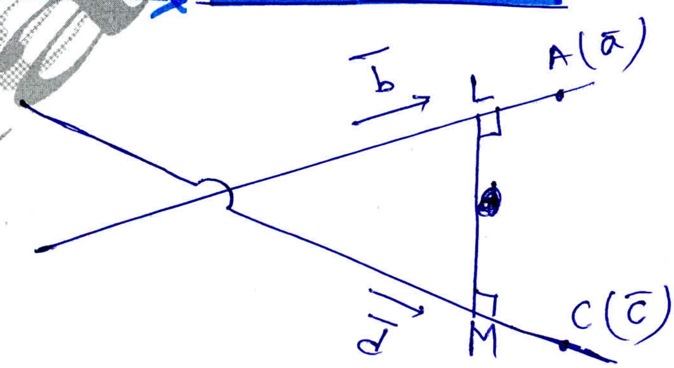
D-10 Shortest Distance between two Skew lines

$\vec{r} = \vec{a} + \lambda \vec{b}$, & $\vec{r} = \vec{c} + t\vec{d}$

Shortest distance

$$LM = \frac{(\vec{a} - \vec{c}) \cdot (\vec{b} \times \vec{d})}{|\vec{b} \times \vec{d}|}$$

$$= \frac{|[\,\vec{a} - \vec{c}\ \ \vec{b}\ \ \vec{d}\,]|}{|\vec{b} \times \vec{d}|}.$$



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Cartesian form

$$\frac{x-a_1}{b_1}=\frac{y-a_2}{b_2}=\frac{z-a_3}{b_3} \;\&\; \frac{x-c_1}{d_1}=\frac{x-c_2}{d_2}=\frac{x-c_3}{d_3}$$

shortest distance $= \dfrac{\begin{vmatrix} c_1-a_1 & c_2-a_2 & c_3-a_3 \\ b_1 & b_2 & b_3 \\ d_1 & d_2 & d_3 \end{vmatrix}}{\begin{vmatrix} i & j & k \\ b_1 & b_2 & b_3 \\ d_1 & d_2 & d_3 \end{vmatrix}}$

Conditions for lines to Intersect

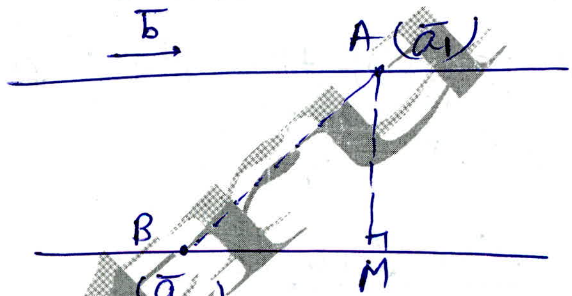
$\bar r=\bar a+\lambda\bar b$ & $\bar r=\bar c+t\bar d$ are intersecting if shortest distance b/w them is zero.

$\Rightarrow$ $[\bar a-\bar c \;\; \bar b \;\; \bar d]=0$ or $\begin{vmatrix} c_1-a_1 & c_2-a_2 & c_3-a_3 \\ b_1 & b_2 & b_3 \\ d_1 & d_2 & d_3 \end{vmatrix}=0$

D-11 Shortest Distance between two Parallel Lines

$\bar r=\bar a_1+\lambda\bar b$ & $\bar r=\bar a_2+t\bar b$

shortest distance $=\left|\dfrac{\bar b\times(\bar a_2-\bar a_1)}{|\bar b|}\right|$



cartesian form

$\bar a_1 = x_1\hat i + x_2\hat j + x_3\hat k$

$\bar a_2 = y_1\hat i + y_2\hat j + y_3\hat k$

$\bar b = a\hat i + b\hat j + c\hat k$

shortest distance $= \dfrac{\begin{vmatrix} \hat i & \hat j & \hat k \\ x_1-y_1 & x_2-y_2 & x_3-y_3 \\ a & b & c \end{vmatrix}}{\sqrt{a^2+b^2+c^2}}$

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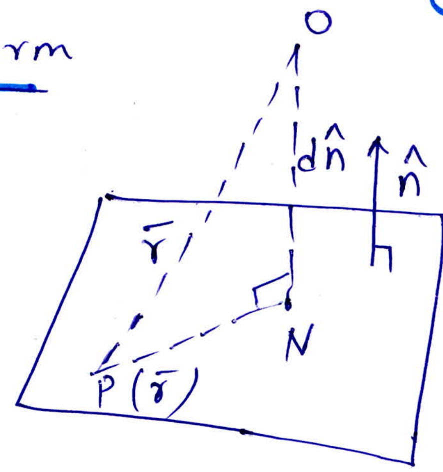
D-12 Eq^n of a Plane in Normal form



$\hat{n}$ is unit normal vector

Plane : $\boxed{\bar{r}\cdot\hat{n} = d}$ vector form

where d is perpendicular distance of plane from origin.

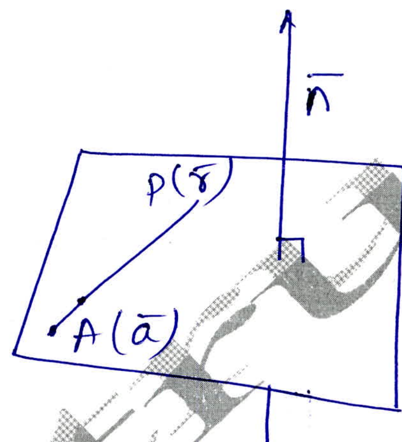
Cartesian form

$\bar{r} = x\hat{i}+y\hat{j}+z\hat{k}$, $\hat{n} = l\hat{i}+m\hat{j}+n\hat{k}$

$\Rightarrow \boxed{lx+my+nz=d}$

If eq^n of plane is $ax+by+cz=d$, then a, b, c are direction ratios of the normal to the plane

D-13. Plane passing thru a given point and Normal to a given vector



$(\bar{r}-\bar{a})\cdot\bar{n} = 0 \Rightarrow \boxed{\bar{r}\cdot\bar{n} = \bar{a}\cdot\bar{n}}$

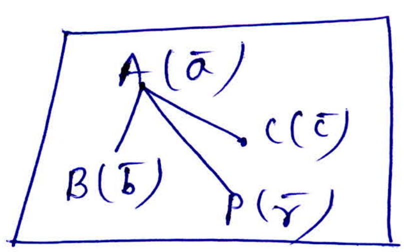
$\perp$ distance of this plane from origin $= \dfrac{\bar{a}\cdot\bar{n}}{|\bar{n}|}$

Cartesian form

$\bar{r} = x\hat{i}+y\hat{j}+z\hat{k}$, $a = x_1\hat{i}+y_1\hat{j}+z_1\hat{k}$, $\bar{n} = a\hat{i}+b\hat{j}+c\hat{k}$

$\Rightarrow \boxed{a(x-x_1)+b(y-y_1)+c(z-z_1)=0}$

D-14 Eq^n of a Plane passing thru three points



$\bar{a}$, $\bar{b}$ & $\bar{c}$ lie on the plane

Hence $\overline{AP}$, $\overline{AB}$ & $\overline{AC}$ are coplanar.

$\Rightarrow [\overline{AP}\ \overline{AB}\ \overline{AC}] = 0$

$\Rightarrow [\bar{r}\ \bar{b}\ \bar{c}] + [\bar{r}\ \bar{a}\ \bar{b}] + [\bar{r}\ \bar{c}\ \bar{a}] = [\bar{a}\ \bar{b}\ \bar{c}]$

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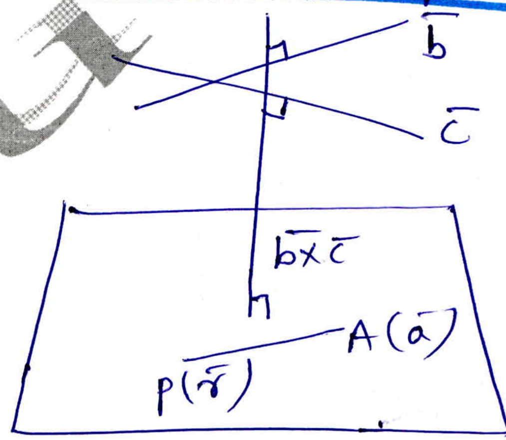
cartesian form

$\bar{a} \equiv (x_1, y_1, z_1) \quad \bar{b} \equiv (x_2, y_2, z_2), \quad \bar{c} \equiv (x_3, y_3, z_3)$

$$\Rightarrow \begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0$$

D-15 Eq^n of a plane passes thru a pt $A(\bar{a})$ and is parallel to given vectors $\bar{b}$ & $\bar{c}$.



$\overline{AP} \cdot (\bar{b} \times \bar{c}) = 0$

$\Rightarrow [\bar{r}\ \bar{b}\ \bar{c}] = [\bar{a}\ \bar{b}\ \bar{c}]$

cartesian form

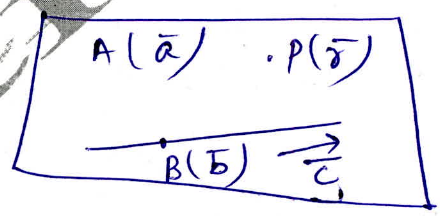
$\bar{b} = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}, \quad \bar{c} = x_3\hat{i} + y_3\hat{j} + z_3\hat{k}$

$\bar{a} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$

$$\Rightarrow \begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = 0$$

D-16 Eq^n of a plane passing thru a given point and a line

line $\bar{r} = \bar{b} + \lambda \bar{c}$, point $\bar{a}$



$\overline{AP}, \overline{AB}, \bar{c}$ are coplanar

$\Rightarrow [\overline{AP}\ \ \overline{AB}\ \ \bar{c}] = 0$

$\Rightarrow [\bar{r} - \bar{a}\ \ \bar{b} - \bar{a}\ \ \bar{c}] = 0$

D-17 Intercept form of a Plane

$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ where a, b, c are x, y & z intercepts of the plane.

D-18 Eq^n of a plane parallel to a given plane

$ax + by + cz + d = 0 \quad \text{—} (1)$

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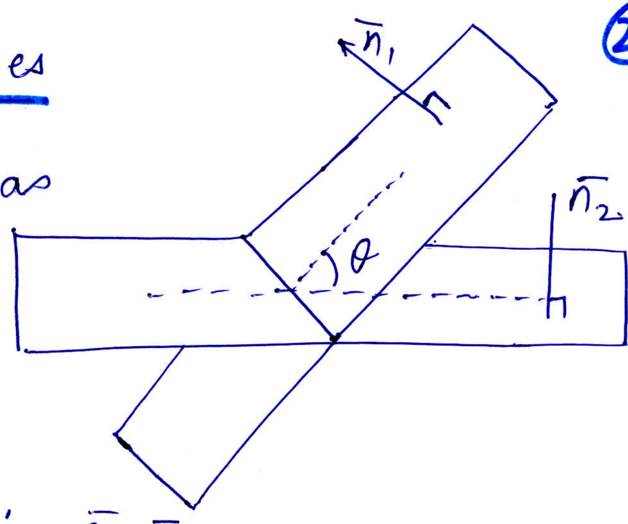
plane $\parallel$ to (1) is $ax + by + cz + k = 0$, where k is any scalar

D-19 Angle between two Planes



angle b/w two planes is same as angle b/w their normals.

$$\cos\theta = \frac{\bar{n}_1 \cdot \bar{n}_2}{|\bar{n}_1|\,|\bar{n}_2|}$$

Condition for perpendicularity : $\bar{n}_1 \cdot \bar{n}_2 = 0$

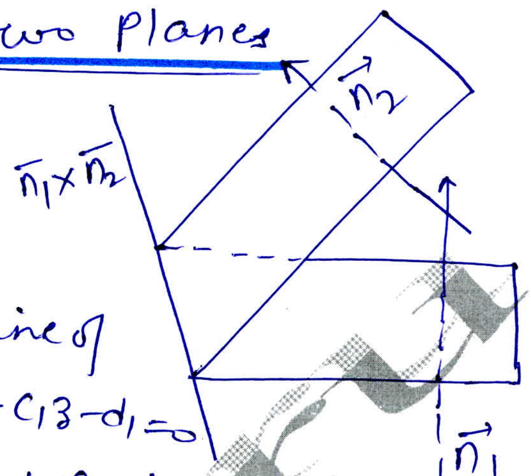
cartesian form

$a_1x + b_1y + c_1z + d_1 = 0$ & $a_2x + b_2y + c_2z + d_2 = 0$

$$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\,\sqrt{a_2^2+b_2^2+c_2^2}}$$

D-20 Line of Intersection of two Planes

$\bar{r}\cdot\bar{n}_1 = d_1$ & $\bar{r}\cdot\bar{n}_2 = d_2$



line of intersection of planes is parallel to vector $\bar{n}_1 \times \bar{n}_2$

If we wish to find the eqⁿ of line of intersection of planes $a_1x + b_1y + c_1z - d_1 = 0$ & $a_2x + b_2y + c_2z - d_2 = 0$, then we find any pt on the line by putting $z = 0$ (say), then we can find corresponding values of x & y by solving $a_1x + b_1y - d_1 = 0$ & $a_2x + b_2y - d_2 = 0$. Thus by fixing the value of $z = \lambda$, we can find the corresponding value of x & y in terms of λ. After getting x, y & z in terms of λ, we can write the eqⁿ of line in symmetric form

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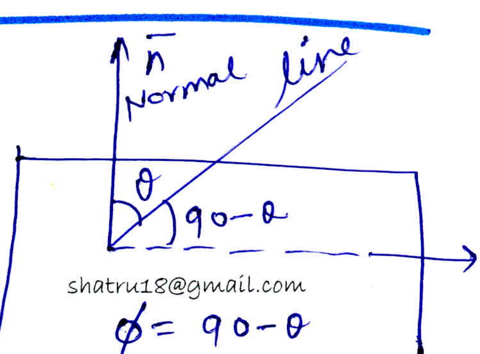
D-21 Angle b/w a Line and a Plane

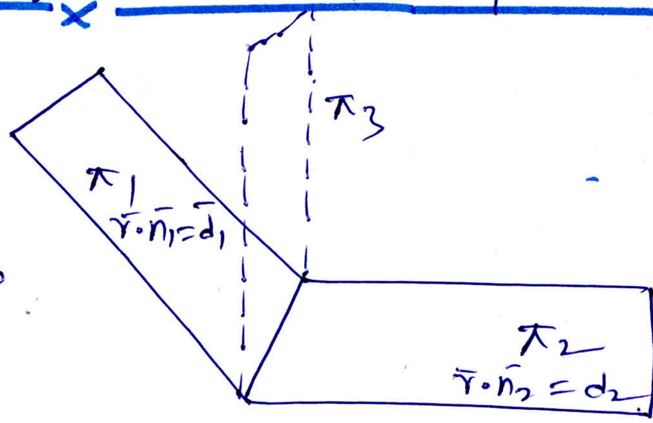
$\bar{r} = \bar{a} + \lambda\bar{b}$, $\bar{r}\cdot\bar{n} = d$



$$\cos\theta = \frac{\bar{b}\cdot\bar{n}}{|\bar{b}|\,|\bar{n}|}$$

$$\sin\phi = \left|\frac{\bar{b}\cdot\bar{n}}{|\bar{b}|\,|\bar{n}|}\right| \Rightarrow \boxed{\phi = \sin^{-1}\left|\frac{\bar{b}\cdot\bar{n}}{|\bar{b}|\,|\bar{n}|}\right|}$$

D-22 Eq^n of a Plane passing thru the line of Intersection of two planes



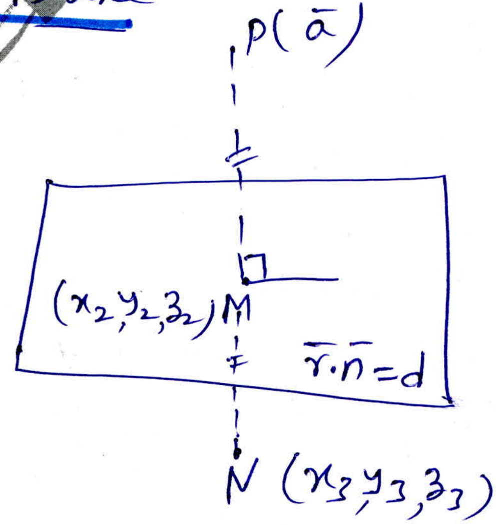
$\bar{r}\cdot\bar{n}_1 = d_1$ & $\bar{r}\cdot\bar{n}_2 = d_2$

Hence the required plane is

$$(\bar{r}\cdot\bar{n}_1 - d_1) + \lambda(\bar{r}\cdot\bar{n}_2 - d_2) = 0$$

$$\Rightarrow \bar{r}\cdot(\bar{n}_1 + \lambda\bar{n}_2) = d_1 + \lambda d_2$$

D-23 Distance of a Point from a Plane



$$PM = \left|\frac{(d - \bar{a}\cdot\bar{n})\,\bar{n}}{|\bar{n}|^2}\right|$$

$$\boxed{PM = \frac{|d - (\bar{a}\cdot\bar{n})|}{|\bar{n}|}}$$

Cartesian form

$P(x_1, y_1, z_1)$, plane $ax + by + cz + d = 0$.

$$PM = \frac{|(ax_1 + by_1 + cz_1 + d)|}{\sqrt{a^2 + b^2 + c^2}}$$

Also coordinates of M are given by

$$\frac{x_2 - x_1}{a} = \frac{y_2 - y_1}{b} = \frac{z_2 - z_1}{c} = -\frac{(ax_1 + by_1 + cz_1 + d)}{a^2 + b^2 + c^2} \quad \text{where } M(x_2, y_2, z_2)$$

If N is the image of P in plane

$$\frac{x_3 - x_1}{a} = \frac{y_3 - y_1}{b} = \frac{z_3 - z_1}{c} = -\frac{2(ax_1 + by_1 + cz_1 + d)}{a^2 + b^2 + c^2}$$

D-24 Distance b/w Parallel Planes

$ax + by + cz + d_1 = 0$ & $ax + by + cz + d_2 = 0$

$$\text{distance} = \left|\frac{d_2 - d_1}{\sqrt{a^2 + b^2 + c^2}}\right|$$

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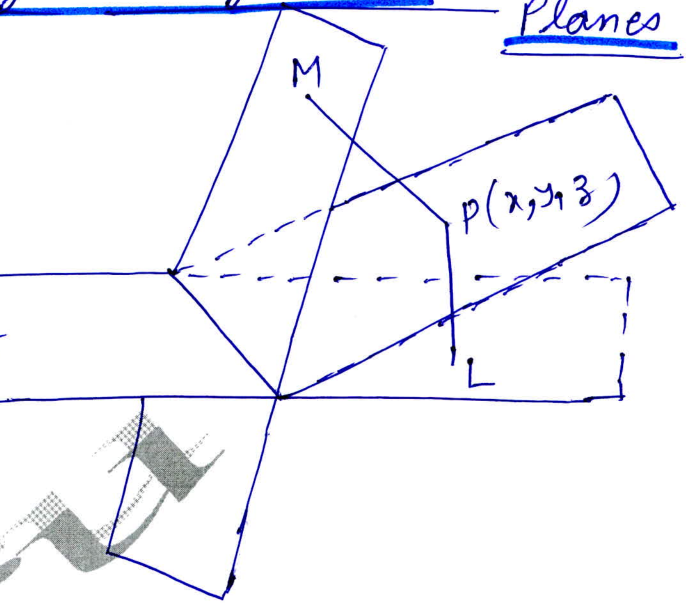
D-25 Eq^n of A Plane Bisecting The Angle Between Two Planes

$a_1x + b_1y + c_1z + d_1 = 0$

$a_2x + b_2y + c_2z + d_2 = 0$

$PL = PM$



$$\Rightarrow \frac{a_1x+b_1y+c_1z+d_1}{\sqrt{a_1^2+b_1^2+c_1^2}} = \pm \frac{a_2x+b_2y+c_2z+d_2}{\sqrt{a_2^2+b_2^2+c_2^2}}$$

Vector form

$$\left|\frac{\bar{r}\cdot\bar{n}_1 - d_1}{|n_1|}\right| = \left|\frac{\bar{r}\cdot\bar{n}_2 - d_2}{|\bar{n}_2|}\right|$$

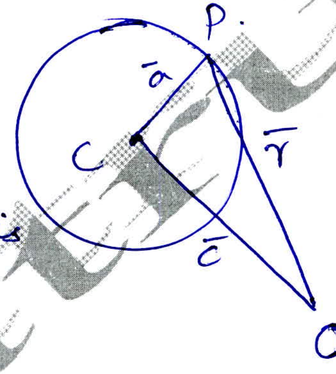
To find the eq^n of bisector ~~plane~~ containing origin or Acute/obtuse angle bisector, or to find the relative position of two points wrt a plane, treat plane as a line and use same concepts as we do in 2-D geometry.

~~Sphere~~

D-26 Sphere

$(\bar{r}-\bar{c})\cdot(\bar{r}-\bar{c}) = a^2$

or $\boxed{|\bar{r}-\bar{c}| = a}$



where $\bar{c}$ is the centre and 'a' is radius

Cartesian form

$\bar{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$, $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$\Rightarrow (x-c_1)^2 + (y-c_2)^2 + (z-c_3)^2 = a^2$

$\Rightarrow x^2+y^2+z^2 - 2c_1x - 2c_2y - 2c_3z + c_1^2 + c_2^2 + c_3^2 - a^2 = 0$

we usually write the eq^n as $x^2+y^2+z^2+2ux+2vy+2wz+d=0$

$\Rightarrow (x+u)^2 + (y+v)^2 + (z+w)^2 = u^2+v^2+w^2-d$

centre $(-u,-v,-w)$ & radius $\sqrt{u^2+v^2+w^2-d^2}$

Sphere with centre as origin $x^2+y^2+z^2=a^2$

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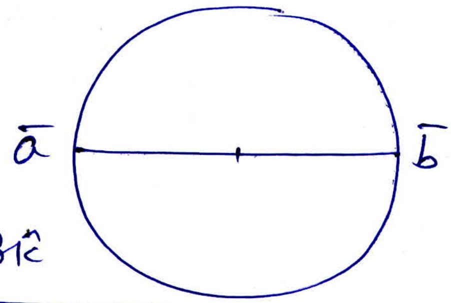
D-27 Diameter form of the Eq^n of a Sphere

$(\bar{r}-\bar{a})\cdot(\bar{r}-\bar{b})=0$



cartesian form

$\bar{a}=x_1\hat{i}+y_1\hat{j}+z_1\hat{k}$

$\bar{b}=x_2\hat{i}+y_2\hat{j}+z_2\hat{k}$, $\bar{r}=x\hat{i}+y\hat{j}+z\hat{k}$

$$\Rightarrow \boxed{(x-x_1)(x-x_2)+(y-y_1)(y-y_2)+(z-z_1)(z-z_2)=0}$$

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